

## **LINKAGES AND MULTIPLIERS IN A MULTIREGIONAL FRAMEWORK: INTEGRATION OF ALTERNATIVE APPROACHES**

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**ABSTRACT:** In this paper, two literatures that have explored the structure of economies are brought together. In the first case, the approaches to key sector identification (initially associated with Hirschman, 1958 and Rasmussen, 1956) that were modified by Cella (1986), Clements and Rossi (1992) and Guilhoto *et al.* (1994) to reveal what may be referred to a pure linkage approach are related to the concerns of Miyazawa (1976) and his identification of internal and external multiplier effects. While Miyazawa was interested mainly in identifying the sources of change in an economy, his approach shares considerable commonality with the new ideas in key sector identification in which a sector or set of sectors are separated from the rest of the economy. Hence, in both cases, a decomposition of the economy needs to be considered; the present paper reveals the similarity of perspective and provides the formal link between the two methodologies.

## **1. INTRODUCTION**

Several important themes focusing on an understanding of the economic structure of economies as represented by input-output systems have appeared recently. First, there has been the recognition that only a small set of transactions or sectors in an economy may be considered to be analytically important, in the sense that changes in their values create significant changes elsewhere in the economy (Sonis and Hewings, 1992, 1995). Secondly, the complexity of transactions in an economy, especially in very detailed interindustry matrices, precludes understanding of the structure of the economy without some translation or decomposition of these transactions to a set of hierarchical flows. As a result, many alternative decompositions have been proposed to assist the analyst in obtaining a better appreciation of the economic structure. It turns out that two separate approaches to these issues share a

methodology that is common in form; in this paper, these methodologies are brought together for the first time and the similarities are explored.

The first method arose from some general dissatisfaction with traditional methods for identifying key sectors, methods initially identified with the work of Hirschman (1958) and Rasmussen (1956). The alternative offered is a procedure to separate out the impacts of a specific sector from the rest of the economy or a single region from the rest of the economy or even a country from the trading bloc in which it is nested. The second method was proposed for an entirely different purpose - the identification of the sources of change in an economy. Here, Miyazawa (1976) attempted to explore the role of internal and external linkages in the propagation of change.

In the next section the previous approaches will be presented. The third section will present a consolidation of the previous approaches, while in the last section some final comments will be made.

## 2. THE PRIOR APPROACHES

Since the Hirschman and Rasmussen indices are well known, they will not be repeated here; attention will focus on the developments initiated by Cella (1984, 1986) and elaborated by Clements (1990), Clements and Rossi (1991, 1992) and Guilhoto *et al.* (1994). Essentially, the approaches may be considered to take the following form: for any sector or set of sectors, extract them from the rest of the economy through a partitioning of the matrix. Through alternative methods of manipulation, an assessment can be made of the role of this sector or set of sectors in the economy as a whole. The differences in contributions focus on different ways in which the extraction method is applied.

### 2.1 The Cella/Clements Approach

Using the Leontief matrix of direct inputs coefficients ( $A$ ), Cella (1984) defined the following block matrices:

$$A = \begin{pmatrix} A_{jj} & A_{jr} \\ A_{rj} & A_{rr} \end{pmatrix} \quad (3)$$

and

$$\bar{A} = \begin{pmatrix} A_{jj} & \mathbf{0} \\ \mathbf{0} & A_{rr} \end{pmatrix} \quad (4)$$

where  $A_{jj}$  and  $A_{rr}$  are square matrices of direct inputs, respectively, within sector  $j$  and within the rest of the economy (economy less sector  $j$ );  $A_{jr}$  and  $A_{rj}$  are rectangular matrices showing, respectively, the direct inputs purchased by sector  $j$  from the rest of the economy and the direct inputs purchased by the rest of the economy from sector  $j$ .  $\bar{A}$  is a matrix of direct input coefficients, defined to confine interaction to those between establishments within sector  $j$  and, similarly, to interaction among the rest of the sectors but excluding  $j$ . A similar perspective

could be applied in a multinational or multiregional economy case in which one nation or region is extracted from the rest (see Sonis *et al.* 1995a, b).

Following Sonis and Hewings (1993), equation (3) can be solved for the Leontief inverse resulting in:

$$L = (I - A)^{-1} = \begin{pmatrix} \tilde{\Delta}_j & \tilde{\Delta}_j A_{jr} \Delta_r \\ \Delta_r A_{rj} \tilde{\Delta}_j & \Delta_r (I + A_{rj} \tilde{\Delta}_j A_{jr} \Delta_r) \end{pmatrix} \quad (5)$$

where:

$$\tilde{\Delta}_j = (I - A_{jj} - A_{jr} \Delta_r A_{rj})^{-1} \quad (6)$$

$$\Delta_r = (I - A_{rr})^{-1} \quad (7)$$

In the same way, equation (4) can be solved for the Leontief inverse yielding:

$$\bar{L} = (I - \bar{A})^{-1} = \begin{pmatrix} \Delta_j & 0 \\ 0 & \Delta_r \end{pmatrix} \quad (8)$$

where:

$$\Delta_j = (I - A_{jj})^{-1} \quad (9)$$

Cella (1984) used this approach to define the total linkage effect ( $TL$ ) of sector  $j$  in the economy, *i.e.*, the difference between the total production in the economy and the production in the economy if sector  $j$  neither bought inputs from the rest of the economy nor sold its output to the rest of the economy. In development terms, this might be regarded as the opposite of import substitution, namely, the disappearance of a whole industrial sector from an economy. Given this assumption, the following definition of  $TL$  may be derived:

$$TL = i' [L - \bar{L}] Y = i' \begin{pmatrix} \tilde{\Delta}_j - \Delta_j & \tilde{\Delta}_j A_{jr} \Delta_r \\ \Delta_r A_{rj} \tilde{\Delta}_j & \Delta_r A_{rj} \tilde{\Delta}_j A_{jr} \Delta_r \end{pmatrix} \begin{pmatrix} Y_j \\ Y_r \end{pmatrix} \quad (10)$$

where  $i'$  is a unit row vector of the appropriate dimension, and  $Y$ ,  $Y_j$  and  $Y_r$  are column vectors of final demand for, respectively, the total economy, sector  $j$  alone, and the rest of economy, excluding sector  $j$ .

Cella (1984) then defined the backward ( $BL$ ) and forward ( $FL$ ) linkage:

$$BL = \left[ (\tilde{\Delta}_j - \Delta_j) + i'_{rr} (\Delta_r A_{rj} \tilde{\Delta}_j) \right] Y_j \quad (11)$$

$$FL = \left[ (\tilde{\Delta}_j A_{jr} \Delta_r) + i'_{rr} (\Delta_r A_{rj} \tilde{\Delta}_j A_{jr} \Delta_r) \right] Y_r \quad (12)$$

where  $i'_{rr}$  is a unit row vector of the appropriate dimension.

Clements argues that the second component of the forward linkage belongs to the backward linkage, as in his words, "it quantifies the stimulus given to

supplying sectors caused by intermediate demand for a given sector" (Clements 1990, p. 339). In that way, he proposed a definition of backward and forward linkage as:

$$BL = \left[ \left( \tilde{\Delta}_j - \Delta_j \right) + i'_{rr} \left( \Delta_r A_{rj} \tilde{\Delta}_j \right) \right] [Y_j] + \left[ i'_{rr} \left( \Delta_r A_{rj} \tilde{\Delta}_j A_{jr} \Delta_r \right) \right] [Y_r] \quad (13)$$

$$FL = \left[ \left( \tilde{\Delta}_j A_{jr} \Delta_r \right) \right] [Y_r] \quad (14)$$

In the next section, some comments about the Cella / Clements technique are provided, and the pure linkage approach is presented.

## 2.2 The Pure Linkage Approach

While, in essence, the idea behind the derivation of Cella / Clement is correct, we think that the application can be improved and the following suggestions are provided. First of all, if one wants to isolate sector  $j$  from the rest of the economy, one should start with the following decomposition<sup>1</sup> as an alternative to that provided in (4).

**Decomposition (I):**

$$A = \begin{pmatrix} A_{jj} & A_{jr} \\ A_{rj} & A_{rr} \end{pmatrix} = \begin{pmatrix} A_{jj} & A_{jr} \\ A_{rj} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_{rr} \end{pmatrix} = A_j + A_r \quad (15)$$

where matrix  $A_j$  represents sector  $j$  isolated from the rest of the economy, and matrix  $A_r$  represents the rest of the economy. As before, define the Leontief inverse as:

$$L = (I - A)^{-1} \quad (16)$$

then it can be shown that each additive decomposition of the matrix of direct inputs (equation 15) can be converted into two alternative multiplicative decomposition of the Leontief inverse as follow (see Sonis and Hewings, 1993):

$$L = P_2 P_1 \quad (17)$$

or

$$L = P_1 P_3 \quad (18)$$

where:

$$P_1 = (I - A_r)^{-1} \quad (19)$$

$$P_2 = (I - P_1 A_j)^{-1} \quad (20)$$

$$P_3 = (I - A_j P_1)^{-1} \quad (21)$$

<sup>1</sup> The four basic types of decomposition to be related together in section 3 will be numbered sequentially in sections 2.2 and 2.3.

Equation (17) isolates the interaction within the rest of the economy ( $P_1$ ) from the interaction of sector  $j$  with the rest of the economy ( $P_2$ ). As can be seen in equation (20),  $P_2$  shows the direct and indirect impacts that the demand for inputs from sector  $j$  will have on the economy ( $P_1 A_j$ ).

Equation (18), on the other hand, isolates the interaction within the rest of the economy ( $P_1$ ) from the interaction of the rest of the economy with sector  $j$  through ( $P_3$ ).  $P_3$  reveals what the level of the impacts on sector  $j$  will be generated by the direct and indirect needs of the rest of the economy ( $A_j P_1$ ).

Working with equations (17), (19), and (20), equation (17) can be expressed in the following form:

$$L = \underbrace{\begin{pmatrix} \tilde{\Delta}_j & \tilde{\Delta}_j A_{jr} \\ \Delta_r A_{rj} \tilde{\Delta}_j & I + \Delta_r A_{rj} \tilde{\Delta}_j A_{jr} \end{pmatrix}}_{P_2} \underbrace{\begin{pmatrix} I & 0 \\ 0 & \Delta_r \end{pmatrix}}_{P_1} \quad (22)$$

where all the variables are defined as before, and the first term on the RHS is  $P_2$  while the second term is  $P_1$ .

From the first term on the RHS of equation (22), the following decomposition can be presented:

$$P_2 = \begin{pmatrix} I & 0 \\ \Delta_r A_{rj} & I \end{pmatrix} \begin{pmatrix} \tilde{\Delta}_j & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & A_{jr} \\ 0 & I \end{pmatrix} \quad (23)$$

where:

$$P_2 = (I - B_j)^{-1} \quad (24)$$

and

$$B_j = P_1 A_j = \begin{pmatrix} A_{jj} & A_{jr} \\ \Delta_r A_{rj} & 0 \end{pmatrix} \quad (25)$$

From equation (25), a *pure backward linkage (PBL)* can be defined as:

$$PBL = i'_{rr} \Delta_r A_{rj} q_{jj} \quad (26)$$

where  $q_{jj}$  is the value of total production in sector  $j$ , and the other variables are as defined before. If one wants to treat sector  $j$  as a sector isolated from the rest of the economy, it is proposed that it will be more appropriate to use the value of total production instead of the value of final demand as used by Cella (1984), given that the vector of total production will work like an vector of final demand of the sector  $j$  on the rest of the economy.

The PBL will give the pure impact on the economy of the value of the total production in sector  $j$ , *i.e.*, the impact that is free from: a) the demand of inputs that sector  $j$  makes from sector  $j$ ; and b) the feedbacks from the economy to sector  $j$  and vice-versa.

Using (18), (19), and (21), equation (18) can be expressed as:

$$L = \underbrace{\begin{pmatrix} I & 0 \\ 0 & \Delta_r \end{pmatrix}}_{P_1} \underbrace{\begin{pmatrix} \tilde{\Delta}_j & \tilde{\Delta}_j A_{jr} \Delta_r \\ \Delta_r A_{rj} \tilde{\Delta}_j & 1 + A_{rj} \tilde{\Delta}_j A_{jr} \Delta_r \end{pmatrix}}_{P_3} \quad (27)$$

where all the variables are as defined before, and the first term on the RHS is  $P_1$  while the second term is  $P_3$ .

From the second term in the RHS of equation (27), the following decomposition can be presented:

$$P_3 = \begin{pmatrix} I & 0 \\ A_{rj} & I \end{pmatrix} \begin{pmatrix} \tilde{\Delta}_j & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & A_{jr} \Delta_r \\ 0 & I \end{pmatrix} \quad (28)$$

where:

$$P_3 = (I - F_j)^{-1} \quad (29)$$

and

$$F_j = A_j P_1 = \begin{pmatrix} A_{jj} & A_{jr} \Delta_r \\ A_{rj} & 0 \end{pmatrix} \quad (30)$$

From equation (30), a *pure forward linkage (PFL)* can be obtained and this is given by:

$$PFL = A_{jr} \Delta_r q_{rr} \quad (31)$$

where  $q_{rr}$  is a column vector of total production in each sector in the rest of the economy. Again, the reason for using the value of total production instead of the value of final demand is the isolation of sector  $j$  from the rest of the economy for the reasons stated above.

The *PFL* will give the pure impact on sector  $j$  of the total production in the rest of the economy. Again, this impact is freed from some of the confusion of definition in the earlier Cella (1984) and Clements (1990) approaches that were noted in the definition of *PBL*.

If one wants to know what the pure total linkage (*PTL*) of each sector is in the economy, for example, to rank them, it is possible to add the *PBL* with the *PFL*, given that these indexes, as defined above, are expressed in actual values rather than as indices. Hence:

$$PTL = PBL + PFL \quad (32)$$

The above derivation is an improvement over the method developed by Cella (1984).

### 2.3 Multiplicative Structure of the Leontief Inverse and the Miyazawa Partitioned Matrix Multiplier

In this section we will be working with the notion of region instead of sector, but, in the same way that in the previous sections one could replace the word sector by the word region, in this section one could easily replace the word region by the word sector.

Consider a two-region input-output system represented by the following block matrix,  $A$ , of direct inputs:

$$A = \begin{pmatrix} A_{jj} & A_{jr} \\ A_{rj} & A_{rr} \end{pmatrix} \quad (33)$$

where  $A_{jj}$  and  $A_{rr}$  are the quadrate matrices of direct inputs within the first and second region and  $A_{jr}$  and  $A_{rj}$  are the rectangular matrices showing the direct inputs purchased by the second region and vice versa. The matrix,  $A$ , can be presented in a separate form, which will be referred to as a "pull-decomposition:" In this perspective, the first region is shown to exert an influence on the second region by pulling inputs (i.e., imports) for production from this second region. A similar perspective applies to the second region's interaction with the first region. Hence, depending upon the perspective employed, the off diagonal entries of (33) may be viewed as "push" or "pull" linkages with the other region.

#### *Decomposition (II):*

$$A = \begin{pmatrix} A_{jj} & 0 \\ A_{rj} & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_{jr} \\ 0 & A_{rr} \end{pmatrix} = A_1 + A_2 \quad (34)$$

If the Leontief inverse exists for the first region, it will be defined as follows (see also equation 9):

$$\Delta_j = (I - A_{jj})^{-1} \quad (35)$$

and, following Miyazawa, this will be referred to as the internal matrix multiplier for the first region.

Consider the block-matrix:

$$G_1 = (I - A_1)^{-1} \quad (36)$$

and, from direct matrix multiplication, the following will be obtained:

$$G_1 = \left( \begin{array}{c|c} \Delta_j & 0 \\ \hline A_{rj}\Delta_j & I \end{array} \right) = \left( \begin{array}{c|c} I & 0 \\ \hline A_{rj} & I \end{array} \right) \left( \begin{array}{c|c} \Delta_j & 0 \\ \hline 0 & I \end{array} \right) \quad (37)$$

Further:

$$G_1(I - A) = G_1[(I - A_1) - A_2] = I - G_1A_2 \quad (38)$$

or:

$$I - G_1 A_2 = \left( \begin{array}{c|c} I & -\Delta_j A_{jr} \\ \hline 0 & I - A_{rr} - A_{rj} \Delta_j A_{jr} \end{array} \right) \quad (39)$$

The Leontief inverse may be defined as:

$$\tilde{\Delta}_r = \left( I - A_{rr} - A_{rj} \Delta_j A_{jr} \right)^{-1} \quad (40)$$

and this is referred to as the external matrix multiplier of the second region revealing the influence of inputs from the first region.<sup>2</sup>

Furthermore, consider the block-matrix:

$$G_2 = \left( I - G_1 A_2 \right) \quad (41)$$

from which, direct matrix multiplication implies that:

$$G_2 = \left( \begin{array}{c|c} I & \Delta_j A_{jr} \tilde{\Delta}_r \\ \hline 0 & \tilde{\Delta}_r \end{array} \right) = \left( \begin{array}{c|c} I & \Delta_j A_{jr} \\ \hline 0 & I \end{array} \right) \left( \begin{array}{c|c} I & 0 \\ \hline 0 & \tilde{\Delta}_r \end{array} \right) \quad (42)$$

Moreover,

$$\begin{aligned} G_2 G_1 (I - A) &= I \text{ or } (I - A)^{-1} = G_2 G_1 = & (43) \\ &= \left( \begin{array}{c|c} I & \Delta_j A_{jr} \\ \hline 0 & I \end{array} \right) \left( \begin{array}{c|c} I & 0 \\ \hline 0 & \tilde{\Delta}_r \end{array} \right) \left( \begin{array}{c|c} I & 0 \\ \hline A_{rj} & I \end{array} \right) \left( \begin{array}{c|c} \Delta_j & 0 \\ \hline 0 & I \end{array} \right) \\ &= \left( \begin{array}{c|c} I & \Delta_j A_{jr} \tilde{\Delta}_r \\ \hline 0 & \tilde{\Delta}_r \end{array} \right) \left( \begin{array}{c|c} \Delta_j & 0 \\ \hline A_{rj} \Delta_j & I \end{array} \right) \end{aligned}$$

In this vision of linkages, each region may be considered to exhibit a self-influence effect (through the standard Leontief influence) and through a push or pull relationship with the other region. Through matrix multiplication, the following Miyazawa formula may be obtained:

$$(I - A)^{-1} = \left( \begin{array}{c|c} \Delta_j + \Delta_j A_{jr} \tilde{\Delta}_r A_{rj} \Delta_j & \Delta_j A_{jr} \tilde{\Delta}_r \\ \hline \tilde{\Delta}_r A_{rj} \Delta_j & \tilde{\Delta}_r \end{array} \right) \quad (44)$$

The multiplicative decomposition (43) presents two important features of regional synergetic interactions. First, each region is featured with a separate block-matrix regional multiplier of identical form and secondly, an hierarchy of interactions are revealed through the regional sub-systems. In this case, for example, the block-matrix of the second region multiplier depends on the influence of the first region on the second region. Obviously, the "order" of the regions is important; if the second region is placed at the top of the hierarchy:

**Decomposition (III):**

$$A = \left( \begin{array}{c|c} 0 & A_{jr} \\ \hline 0 & A_{rr} \end{array} \right) + \left( \begin{array}{c|c} A_{jj} & 0 \\ \hline A_{rj} & 0 \end{array} \right) = A'_1 + A'_2 \quad (45)$$

<sup>2</sup> This terminology and interpretation is different from the original definitions in Miyazawa's work.



then:

$$\begin{aligned} G_1' &= (I - A_1')^{-1} = \left( \begin{array}{c|c} I & -A_{jr} \\ \hline 0 & I - A_{rr} \end{array} \right)^{-1} = \\ &= \left( \begin{array}{c|c} I & A_{jr}\Delta_r \\ \hline 0 & \Delta_r \end{array} \right) = \left( \begin{array}{c|c} I & A_{jr} \\ \hline 0 & I \end{array} \right) \left( \begin{array}{c|c} I & 0 \\ \hline 0 & \Delta_r \end{array} \right) \end{aligned} \quad (46)$$

where (see also equation 7)  $\Delta_r = (I - A_{rr})^{-1}$  is the internal matrix multiplier for the second region. Further,

$$\begin{aligned} G_2' &= (I - G_1'A_2')^{-1} = \left( \begin{array}{c|c} I - A_{jj} - A_{jr}\Delta_r A_{rj} & 0 \\ \hline -\Delta_r A_{rj} & I \end{array} \right)^{-1} = \\ &= \left( \begin{array}{c|c} \tilde{\Delta}_j & 0 \\ \hline \Delta_r A_{rj} \tilde{\Delta}_j & I \end{array} \right) = \left( \begin{array}{c|c} I & 0 \\ \hline \Delta_r A_{rj} & I \end{array} \right) \left( \begin{array}{c|c} \tilde{\Delta}_j & 0 \\ \hline 0 & I \end{array} \right) \end{aligned} \quad (47)$$

where, as also defined in equation (6),

$$\tilde{\Delta}_j = (I - A_{jj} - A_{jr}\Delta_r A_{rj})^{-1} \quad (48)$$

is the external multiplier for the first region as it is influenced now by the second region. Furthermore,

$$\begin{aligned} (I - A)^{-1} &= G_2'G_1' = \\ &= \left( \begin{array}{c|c} I & 0 \\ \hline \Delta_r A_{rj} & I \end{array} \right) \left( \begin{array}{c|c} \tilde{\Delta}_j & 0 \\ \hline 0 & I \end{array} \right) \left( \begin{array}{c|c} I & A_{jr} \\ \hline 0 & I \end{array} \right) \left( \begin{array}{c|c} I & 0 \\ \hline 0 & \Delta_r \end{array} \right) = \\ &= \left( \begin{array}{c|c} \tilde{\Delta}_j & 0 \\ \hline \Delta_r A_{rj} \tilde{\Delta}_j & I \end{array} \right) \left( \begin{array}{c|c} I & A_{jr}\Delta_r \\ \hline 0 & \Delta_r \end{array} \right) = \left( \begin{array}{c|c} \tilde{\Delta}_j & \tilde{\Delta}_j A_{jr}\Delta_r \\ \hline \Delta_r A_{rj} \tilde{\Delta}_j & \Delta_r + \Delta_r A_{rj} \tilde{\Delta}_j A_{jr}\Delta_r \end{array} \right) \end{aligned} \quad (49)$$

which reveals another version of the Miyazawa formula provided in (44). Essentially, (49) corresponds to the same set of regional sub-systems but with a transformation of the hierarchical arrangement of the regions.

A comparison of the components of the equations (44) and (49) yields the following equalities:

$$\begin{aligned} \tilde{\Delta}_j &= \Delta_j + \Delta_j A_{jr} \tilde{\Delta}_r A_{rj} \Delta_j; & \Delta_j A_{jr} \tilde{\Delta}_r &= \tilde{\Delta}_j A_{jr} \Delta_r \\ \tilde{\Delta}_r &= \Delta_r + \Delta_r A_{rj} \tilde{\Delta}_j A_{jr} \Delta_r; & \Delta_r A_{rj} \tilde{\Delta}_j &= \tilde{\Delta}_r A_{rj} \Delta_j \end{aligned} \quad (50)$$

Consider further, the following additive decomposition of the matrix of direct inputs:

**Decomposition (IV):**

$$A = \left( \begin{array}{c|c} A_{jj} & 0 \\ \hline 0 & A_{rr} \end{array} \right) + \left( \begin{array}{c|c} 0 & A_{jr} \\ \hline A_{rj} & 0 \end{array} \right) = A_1'' + A_2'' \quad (51)$$

This decomposition represents the hierarchy of two sub-systems; the matrix,  $A_1''$ , corresponds to the intra-region (domestic) inputs in the two regions and the matrix,  $A_2''$ , captures the system of interregional inputs.

Consider the matrix:

$$G_1'' = (I - A_1'')^{-1} = \left( \begin{array}{c|c} \Delta_j & 0 \\ \hline 0 & \Delta_r \end{array} \right) \quad (52)$$

and the matrix:

$$G_2'' = (I - G_1'' A_2'')^{-1} = \left( \begin{array}{c|c} I & -\Delta_j A_{jr} \\ \hline -\Delta_r A_{rj} & I \end{array} \right)^{-1} \quad (53)$$

The application of (44) and substituting  $A_{jj}$  and  $A_{rr}$  by zero matrices and  $A_{jr}$ ,  $A_{rj}$  by  $\Delta_j A_{jr}$ ,  $\Delta_r A_{rj}$  yields:

$$G_2'' = \left( \begin{array}{c|c} I + \Delta_j A_{jr} \Delta_{rr} \Delta_r A_{rj} & \Delta_j A_{jr} \Delta_{rr} \\ \hline \Delta_{rr} \Delta_r A_{rj} & \Delta_{rr} \end{array} \right) \quad (54)$$

where:

$$\Delta_{rr} = (I - \Delta_r A_{rj} \Delta_j A_{jr})^{-1} \quad (55)$$

may be interpreted as the Miyazawa external matrix multiplier for the second region. Therefore:

$$(I - A)^{-1} = G_2'' G_1'' = \left( \begin{array}{c|c} \Delta_j + \Delta_j A_{jr} \Delta_{rr} \Delta_r A_{rj} \Delta_j & \Delta_j A_{jr} \Delta_{rr} \Delta_r \\ \hline \Delta_{rr} \Delta_r A_{rj} \Delta_j & \Delta_{rr} \Delta_r \end{array} \right) \quad (56)$$

This presentation actually reflects the following hierarchy of three sub-systems, corresponding to the additive decomposition:

$$A = \left( \begin{array}{c|c} A_{jj} & 0 \\ \hline 0 & A_{rr} \end{array} \right) + \left( \begin{array}{c|c} 0 & 0 \\ \hline A_{rj} & 0 \end{array} \right) + \left( \begin{array}{c|c} 0 & A_{jr} \\ \hline 0 & 0 \end{array} \right) \quad (57)$$

A comparison of (56) with (44) provides:

$$\tilde{\Delta}_r = \Delta_{rr} \Delta_r \quad (58)$$

which corresponds to Miyazawa's formulation. It may be interpreted as follows: the external matrix multiplier of the second region under the influence of inputs from the first region equals the internal multiplier of the second region pre-multiplied by the external matrix multiplier of the second region.

Another form of the matrix  $G_2''$  is given by:

$$G_2'' = \left( \begin{array}{c|c} \Delta_{jj} & \Delta_{jj}\Delta_j A_{jr} \\ \hline \Delta_r A_{rj} \Delta_{jj} & I + \Delta_r A_{rj} \Delta_{jj} \Delta_j A_{jr} \end{array} \right) \quad (59)$$

where:

$$\Delta_{jj} = (I - \Delta_j A_{jr} \Delta_r A_{rj})^{-1} \quad (60)$$

is the Miyazawa external multiplier of the first region.

From this form, a multiplicative decomposition may be obtained:

$$(I - A)^{-1} = \left( \begin{array}{c|c} \Delta_{jj}\Delta_j & \Delta_{jj}\Delta_j A_{jr}\Delta_r \\ \hline \Delta_r A_{rj} \Delta_{jj} \Delta_j & I + \Delta_r A_{rj} \Delta_{jj} \Delta_j A_{jr} \Delta_r \end{array} \right) \quad (61)$$

which corresponds to an hierarchy obtained from the decomposition:

$$A = \left( \begin{array}{c|c} A_{jj} & 0 \\ \hline 0 & A_{rr} \end{array} \right) + \left( \begin{array}{c|c} 0 & A_{jr} \\ \hline 0 & 0 \end{array} \right) + \left( \begin{array}{c|c} 0 & 0 \\ \hline A_{rj} & 0 \end{array} \right) \quad (62)$$

The comparison of (61) and (49) yields:

$$\tilde{\Delta}_j = \Delta_{jj}\Delta_j \quad (63)$$

which may be interpreted as the external multipliers of the first region under the influence of the inputs from the second region and is equal to the internal multiplier of the first region pre-multiplied by the external multiplier for the first region.

Using (56) and (61) the following may be obtained:

$$\begin{aligned} (I - A)^{-1} &= \left( \begin{array}{c|c} \Delta_{jj}\Delta_j & \Delta_{jj}\Delta_j A_{jr}\Delta_r \\ \hline \Delta_{rr}\Delta_r A_{rj}\Delta_j & \Delta_{rr}\Delta_r \end{array} \right) = \\ &= \left( \begin{array}{c|c} \Delta_{jj} & 0 \\ \hline 0 & \Delta_{rr} \end{array} \right) \left( \begin{array}{c|c} I & \Delta_j A_{jr} \\ \hline \Delta_r A_{rj} & I \end{array} \right) \left( \begin{array}{c|c} \Delta_j & 0 \\ \hline 0 & \Delta_r \end{array} \right) \end{aligned} \quad (64)$$

which multiplicatively separates the Miyazawa internal and external, intraregional multipliers from the interregional effects. In terms of the system developed by Miller (1966, 1969), the first two matrices of (64) were combined and referred to as the interregional feedback effects. The advantage of (64) in this form is the separation of these feedback effects into external and push or pull effects.

### 3. CONSOLIDATION OF PREVIOUS APPROACHES

As it was presented in the previous section, from the following block matrix:

$$A = \begin{pmatrix} A_{jj} & A_{jr} \\ A_{rj} & A_{rr} \end{pmatrix} \quad (65)$$

four alternative decompositions have been proposed (equations 15, 34, 45, and 51):

$$\text{Decomposition (I):} \quad A = \begin{pmatrix} A_{jj} & A_{jr} \\ A_{rj} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_{rr} \end{pmatrix} = A_j + A_r$$

$$\text{Decomposition (II):} \quad A = \begin{pmatrix} A_{jj} & 0 \\ A_{rj} & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_{jr} \\ 0 & A_{rr} \end{pmatrix} = A_1 + A_2$$

$$\text{Decomposition (III):} \quad A = \left( \begin{array}{c|c} 0 & A_{jr} \\ \hline 0 & A_{rr} \end{array} \right) + \left( \begin{array}{c|c} A_{jj} & 0 \\ \hline A_{rj} & 0 \end{array} \right) = A_1' + A_2'$$

$$\text{Decomposition (IV):} \quad A = \left( \begin{array}{c|c} A_{jj} & 0 \\ \hline 0 & A_{rr} \end{array} \right) + \left( \begin{array}{c|c} 0 & A_{jr} \\ \hline A_{rj} & 0 \end{array} \right) = A_1'' + A_2''$$

The presentation to be followed will show that it is possible to combine the results derived from the 4 previous decompositions.

From equation (65) one can arrive to:

$$B = (I - A)^{-1} = \begin{pmatrix} B_{jj} & B_{jr} \\ B_{rj} & B_{rr} \end{pmatrix} = \begin{pmatrix} \Delta_{jj} & 0 \\ 0 & \Delta_{rr} \end{pmatrix} \begin{pmatrix} \Delta_j & 0 \\ 0 & \Delta_r \end{pmatrix} \begin{pmatrix} I & A_{jr}\Delta_r \\ A_{rj}\Delta_j & I \end{pmatrix} \quad (66)$$

And all the components were defined before, i.e.:

$$\Delta_j = (I - A_{jj})^{-1} \quad \text{Eq. (9) and Eq. (35)}$$

$$\Delta_r = (I - A_{rr})^{-1} \quad \text{Eq.(7) and Eq. (46)}$$

$$\Delta_{jj} = (I - \Delta_j A_{jr} \Delta_r A_{rj})^{-1} \quad \text{Eq. (60)}$$

$$\Delta_{rr} = (I - \Delta_r A_{rj} \Delta_j A_{jr})^{-1} \quad \text{Eq. (55)}$$

In that way, from equation (66) it is possible to see how the process of production occurs in the economy as well as derive a set of multipliers/linkages.

As it was defined above, the matrix

$$\begin{pmatrix} \Delta_{jj} & 0 \\ 0 & \Delta_{rr} \end{pmatrix} \quad (67)$$

can be interpreted as the Miyazawa (1976) external multipliers for region  $j$  and the rest of the economy,  $r$ . And, the matrix

$$\begin{pmatrix} \Delta_j & 0 \\ 0 & \Delta_r \end{pmatrix} \quad (68)$$

can be interpreted as the Miyazawa (1976) internal multipliers for region  $j$  and the rest of the economy,  $r$ . In the matrix

$$\begin{pmatrix} I & A_{jr}\Delta_r \\ A_{rj}\Delta_j & I \end{pmatrix} \quad (69)$$

the first row separates the final demand by its origin, i.e., distinguishes between the final demand that comes from inside the region ( $I$ ) from the one that comes from outside the region ( $A_{jr}\Delta_r$ ). The same idea applies to the second row.

From the Leontief formulation:

$$X = (I - A)^{-1}Y \quad (70)$$

and using the information contained in equations (66) through (69) one can derive a set of indexes that can be used: a) to rank the regions in terms of its importance in the economy; b) to see how the production process occurs in the economy. From equations (66) and (70) one gets:

$$\begin{pmatrix} X_j \\ X_r \end{pmatrix} = \begin{pmatrix} \Delta_{jj} & 0 \\ 0 & \Delta_{rr} \end{pmatrix} \begin{pmatrix} \Delta_j & 0 \\ 0 & \Delta_r \end{pmatrix} \begin{pmatrix} I & A_{jr}\Delta_r \\ A_{rj}\Delta_j & I \end{pmatrix} \begin{pmatrix} Y_j \\ Y_r \end{pmatrix} \quad (71)$$

which leads to:

$$\begin{pmatrix} X_j \\ X_r \end{pmatrix} = \begin{pmatrix} \Delta_{jj} & 0 \\ 0 & \Delta_{rr} \end{pmatrix} \begin{pmatrix} \Delta_j & 0 \\ 0 & \Delta_r \end{pmatrix} \begin{pmatrix} Y_j + A_{jr}\Delta_r Y_r \\ A_{rj}\Delta_j Y_j + Y_r \end{pmatrix} \quad (72)$$

where

$$A_{jr}\Delta_r Y_r \quad (73)$$

is the direct impact of the rest of the economy final demand on region  $j$ , i.e., it gives the level of exports in region  $j$  that are needed to satisfy the production necessities of rest of the economy for a level of final demand given by  $Y_r$ ; and

$$A_{rj}\Delta_j Y_j \quad (74)$$

is the direct impact of region  $j$  final demand on the rest of the economy, i.e., it gives the level of exports in rest of the economy that are needed to satisfy the production necessities of region  $j$  for a level of final demand given by  $Y_j$ .

Continuing from equation (72):

$$\begin{pmatrix} X_j \\ X_r \end{pmatrix} = \begin{pmatrix} \Delta_{jj} & 0 \\ 0 & \Delta_{rr} \end{pmatrix} \begin{pmatrix} \Delta_j Y_j + \Delta_j A_{jr}\Delta_r Y_r \\ \Delta_r A_{rj}\Delta_j Y_j + \Delta_r Y_r \end{pmatrix} \quad (75)$$

One has new definitions for the Pure Backward Linkage (PBL) and for the Pure Forward Linkage (PFL), i.e.,

$$\begin{aligned} PBL &= \Delta_r A_{rj}\Delta_j Y_j \\ PFL &= \Delta_j A_{jr}\Delta_r Y_r \end{aligned} \quad (76)$$

where the PBL will give the pure impact on the rest of the economy of the value of the total production in region  $j$ ,  $(\Delta_j Y_j)$ : i.e., the impact that is free from a)

the demand inputs that region  $j$  makes from region  $j$ , and b) the feedbacks from the rest of the economy to region  $j$  and vice-versa. The PFL will give the pure impact on region  $j$  of the total production in the rest of the economy  $(\Delta_r Y_r)$ .

Continuing from equation (75):

$$\begin{pmatrix} X_j \\ X_r \end{pmatrix} = \begin{pmatrix} \Delta_{jj} \Delta_j Y_j + \Delta_{jj} \Delta_j A_{jr} \Delta_r Y_r \\ \Delta_{rr} \Delta_r A_{rj} \Delta_j Y_j + \Delta_{rr} \Delta_r Y_r \end{pmatrix} = \begin{pmatrix} X_j^j + X_j^r \\ X_r^j + X_r^r \end{pmatrix} \quad (77)$$

so, the level of total production in region  $j$  can be broken down into two components:

$$\begin{aligned} X_j^j &= \Delta_{jj} \Delta_j Y_j \\ X_j^r &= \Delta_{jj} \Delta_j A_{jr} \Delta_r Y_r \end{aligned} \quad (78)$$

where the first component,  $X_j^j$ , gives the level of total production in region  $j$  that is due to the level of final demand in region  $j$ , and the second component,  $X_j^r$ , will give the level of total production in region  $j$  that is due to the level of final demand in the rest of the economy.

In the same way, the level of total production in rest of the economy can also be broken down into two components:

$$\begin{aligned} X_r^j &= \Delta_{rr} \Delta_r A_{rj} \Delta_j Y_j \\ X_r^r &= \Delta_{rr} \Delta_r Y_r \end{aligned} \quad (79)$$

where the first component,  $X_r^j$ , gives the level of total production in rest of the economy that is due to the level of final demand in region  $j$ , and the second component,  $X_r^r$ , will give the level of total production in the rest of the economy that is due to the level of final demand in the rest of the economy.

#### 4. CONCLUSION

The main contribution of this paper was to show, using different matrixes decompositions, a formal link between two different approaches: one directed to the identification of key sectors; the other directed to identifying the sources of change in an economy. In this way, with the new development it is possible to break-down the impact of a sector/region in the economy on its various components.

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